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M. A./M. Sc. (First Semester)

EXAMINATION, 2024-25

MATHEMATICS

Paper Fourth

(Complex Analysis)

(MATH—004)

Time : Two hours] **[Maximum Marks : 60**

Note : *Attempt any **four** questions. Each question carries 15 marks.*

1. (a) If the function $f(z)$ is analytic when $|z| < R$ and has Taylor's expansion $\sum_{0}^{\infty} a_n z^n$, show that if $r < R$:

$$\frac{1}{2\pi}\int_0^{2\pi}\left|f\left(re^{i\theta}\right)\right|^2 d\theta = \sum_{n=0}^{\infty}\left|a_n\right|^2 r^{2n}$$

(P.T.O.)

(b) Discuss the application of the transformation :

$$w = \frac{iz + 1}{z + i}$$

to the area in the z –plane which are respectively inside and outside the unit circle with its centre at the origin.

2. (a) If λ is real, a, b are complex numbers such that $|a| > |b|$, show that the bilinear transformation :

$$w = e^{i\lambda} \frac{az + b}{\bar{a} + \bar{b}z}$$

maps the inside of the circle $|z| = 1$ on the inside or the circle $|w| = 1$.

(b) Prove that

$$e^{1/2c\left(z - \frac{1}{z}\right)} = \sum_{n=-\infty}^{\infty} a_n z^n$$

where $a_n = \frac{1}{2\pi}\int_0^{2\pi} \cos(n\theta - c\sin\theta)\,d\theta$.

3. (a) Show that the relation :

$$w = \frac{5 - 4z}{4z - 2}$$

transform the circle $|z| = 1$ into a circle of radius unity in the w-plane and find the centre of this circle.

(b) If $f(z)$ be an integral function satisfying the inequality $|f(z)| \leq M$ for all values of z where M is a positive constant, then prove that $f(z)$ is constant.

4.(a) Prove that the function which has no singularity in the finite part of the plane or at infinity is constant.

(b) Find the kind of singularities of the following functions :

(i) $\tan\left(-\frac{1}{z}\right)$ at $z = 0$

(ii) z cosec z at $z = \infty$

(iii) at $\frac{\pi z}{(z-a)^2}$ at $z = 0$ and $z = \infty$

5.(a) Find the general bilinear transformation which transforms the unit circular disc $|z| < 1$ onto the unit circular disc $|w| \leq 1$.

(b) Find the bilinear transformation which maps $z = 1, i, -1$ respectively onto $w = i, 0, -i$. For this transformation, find the images of :

(i) $|z| \leq 1$

(ii) Concentric circle $|z| = r, (r > 1)$

6. (a) Applying Calculus of residues, evaluate the following integral :

$$\int_0^{2\pi} \frac{\sin^2\theta \, d\theta}{a + b\cos\theta}; \; a > b > 0$$

(b) Evaluate the integral :

$$\int_c \frac{dz}{(z-3)(z^{13}-1)}; \; c \text{ is } |z| = 2$$

using Cauchy's residues theorem.

7. Evaluate the following by the method of contour integration :

(a) $\int_0^{2\pi} \frac{d\theta}{1 + a^2 - 2a\cos\theta}$, where $0 < a < 1$.

(b) $\int_0^{\infty} \frac{\cos x \, dx}{a^2 - x^2}$.

8. (a) Using Mittag – Leffler : theorem, prove that :

$$\frac{\pi^2}{\sin^2 \pi^2} = \sum_{n=-\infty}^{\infty} - \frac{1}{(z-n)^2}$$

(b) Let the function $f(z)$ be analytic in $r_1 \le |z| \le r_3$ and let $r_1 < r_2 < r_3$ If M_1, M_2, M_3 are respectively the maxima of $|f(z)|$ on the three circles $|z| = r_1, r_2, r_3$ then :

$$M_2^{\log(r_3/r_1)} \le M_3^{\log(r_2/r_1)} \; M_1^{\log(r_3/r_2)}$$